Proper connection number and connected dominating sets*

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Abstract

The proper connection number $pc(G)$ of a connected graph G is defined as the minimum number of colors needed to color its edges, so that every pair of distinct vertices of G is connected by at least one path in G such that no two adjacent edges of the path are colored the same, and such a path is called a proper path. In this paper, we show that for every connected graph with diameter 2 and minimum degree at least 2, its proper connection number is 2. Then, we give an upper bound $\frac{3n}{\delta+1} - 1$ for every connected graph of order n and minimum degree δ. We also show that for every connected graph G with minimum degree at least 2, the proper connection number $pc(G)$ is upper bounded by $pc(G[D]) + 2$, where D is a connected two-way (two-step) dominating set of G. Bounds of the form $pc(G) \leq 4$ or $pc(G) = 2$, for many special graph classes follow as easy corollaries from this result, which include connected interval graphs, asteroidal triple-free graphs, circular arc graphs, threshold graphs and chain graphs, all with minimum degree at least 2. Furthermore, we get the sharp upper bound 3 for the proper connection numbers of interval graphs and circular arc graphs through analyzing their structures.

Keywords: proper connection number; proper-path coloring; connected dominating set; diameter

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1 Introduction

All graphs in this paper are finite, connected and simple. An *edge-coloring* of a graph is a mapping from its edge set to the set of natural numbers. A path in an edge-colored graph with no two edges sharing the same color is called a *rainbow path*. An edge-colored graph G is said to be *rainbow connected* if every pair of distinct vertices of G is connected by at least one rainbow path in G. Such a coloring is called a *rainbow coloring* of the graph. The concept of rainbow coloring was introduced by Chartrand et al. in [5]. Since then, many researchers have been studied the problem on the rainbow connection and got many nice results, see [6, 8, 9] for examples. For more details we refer to a survey paper [7] and a book [8].

Inspired by rainbow coloring and proper coloring in graphs, Andrews et al. [1] introduced the concept of proper-path coloring. Let G be an edge-colored graph. A path P in G is called a *proper path* if no two adjacent edges of P are colored the same. An edge-coloring c is a *proper-path coloring* of a connected graph G if every pair of distinct vertices u, v of G are connected by a proper u-v path in G. If k colors are used, then c is referred to as a *proper-path k-coloring*. The minimum number of colors needed to produce a proper-path coloring of G is called the *proper connection number* of G, denoted by $pc(G)$. Form the definition, it is easy to check that $pc(G) = 1$ if and only if $G = K_n$ and $pc(G) = m$ if and only if $G = K_{1,m}$. For more results, we refer to [1, 3].

A *dominating set* for a graph $G = (V, E)$ is a subset D of V such that every vertex not in D is adjacent to at least one member of D. The number of vertices in a smallest dominating set for G is called the *domination number*, denoted by $\gamma(G)$. The dominating set is every useful to determine some relationship between a subgraph and its supergraph. There are many generalized dominating sets, which will be introduced in the following section and considered in this paper.

We will use two-way dominating sets or a two-way two-step dominating sets of a graph G to help us find upper bounds of the proper connection number $pc(G)$. In Section 2, some definitions and properties of the proper connection number of a graph are given. In Section 3, we give the bound $pc(G) \le pc(G[D]) + 2$, where D is a connected two-way two-step dominating set of G. And we get the following two results as its corollaries: One is that the proper connection number of a chain graph with minimum degree at least 2 is 2; the other is that for every connected graph of order n and minimum degree δ, its proper connection number is upper bounded by $\frac{3n}{\delta+1} - 1$. In addition, we also get that a graph with diameter 2 and minimum degree at least 2 has proper connection number 2. In Section 4, we turn to using connected two-way dominating sets D of G. The inequality $pc(G) \le pc(G[D]) + 2$ and upper bounds for interval graphs, asteroidal triple-free graphs,

circular arc graphs and threshold graphs are obtained. Furthermore, we get the sharp upper bound 3 for proper connection numbers of interval graphs and circular arc graphs through analyzing their structures.

2 Preliminaries

In this section, we introduce some definitions and present several useful facts about the path connection numbers of graphs. We begin with some basic conceptions.

Definition 2.1. Let G be a connected graph. The *distance between two vertices* u and v in G, denoted by $d(u,v)$, is the length of a shortest path between them in G. The *eccentricity* of a vertex v is $ecc(v) := max_{x\in V(G)}d(v,x)$. The *diameter* of G is $diam(G) := max_{x\in V(G)}ecc(x)$. The *radius* of G is $rad(G) := min_{x\in V(G)}ecc(x)$. The *distance between a vertex v and a set $S \subseteq V(G)$* is $d(v,S) := min_{x\in S}d(v,x)$. The *k-step neighborhood* of a set $S \subseteq V(G)$ is $N^k(S) := \{x \in V(G)|d(x,S) = k\}, k \in \{0,1,2,\ldots\}$. The *degree* of a vertex v is $deg(v) := |N^1(v)|$. The *minimum degree* of G is $\delta(G) := min_{x\in V(G)}deg(x)$. A vertex is called *pendant* if its degree is 1 and *isolated* if its degree is 0. We may use $N^k(v)$ in place of $N^k(\{v\})$.

Definition 2.2. Given a graph G, a set $D \subseteq V(G)$ is called a *k-step dominating set* of G, if every vertex in G is at a distance at most k from D. Further, if D induces a connected subgraph of G, it is called a *connected k-step dominating set* of G.

Definition 2.3. A dominating set D in a graph G is called a *two-way dominating set*, if every pendant vertex of G is included in D. In addition, if $G[D]$ is connected, we call D a *connected two-way dominating set*.

Definition 2.4. A two-step dominating set D of vertices in a graph G is called a *two-way two-step dominating set* if

(i) every pendant vertex of G is included in D and

(ii) every vertex in $N^2(D)$ has at least two neighbors in $N^1(D)$.

Further, if $G[D]$ is connected, D is called a —it connected two-way two-step dominating set of G.

Definition 2.5. Let G be a graph and s a positive integer. Define sG as disjoint union of s copies of the graph G, i.e., $sG = \underbrace{G \cup G \cup \cdots \cup G}_{s}$.

Definition 2.6. A *Hamiltonian path* in a graph G is a path containing every vertex of G. And a graph having a Hamiltonian path is called a *traceable graph*.

We state some known simple results on the proper-path coloring and two-way two-step dominating set which will be useful in the sequel..

Lemma 2.1. *If P is a path, then $pc(P) = 2$.*

Lemma 2.2. *[1] If G is a traceable graph that is not complete, then $pc(G) = 2$.*

Proposition 2.1. [1] If T be a nontrivial tree, then $pc(T) = \chi' = \Delta$.

In the same paper [1], there is a lemma which will be useful in the following proof.

Lemma 2.3. *[1] If G is a nontrivial connected graph and H is a connected spanning subgraph of G, then $pc(G) \leq pc(H)$. In particular, $pc(G) \leq pc(T)$ for every spanning tree T of G.*

3 Proper connection number and connected two-way two-step dominating set

In this section, we will give an upper bound of proper connection number of a graph G by using the connected two-way two-step dominating sets.

Let D be a connected two-way two-step dominating set of a graph G. This implies that every vertex $v \in V(G) \setminus D$ has two edge-disjoined paths connecting to D. Our idea is to color $G[D]$ first and then to color all the other edges with a constant number of colors, ensuring a proper-path coloring of a graph G. Now we give our main theorems.

Theorem 3.1. *If D is a connected two-way two-step dominating set of a graph G, then*

$$pc(G) \leq pc(G[D]) + 2.$$

Proof. Let H be a spanning subgraph of a graph G. Then $pc(G) \leq pc(H)$ by Lemma 2.3. In the following, we will give a proper-path coloring of H with $pc(G[D]) + 2$ colors, and then prove the theorem.

Let c_D be a proper-path coloring of $G[D]$ using colors $\{3, 4, \cdots, k := pc(G[D]), k+1, k+2\}$. For $x \in N^1(D)$, a neighbor of x in D is called a foot of x. Define the set of foots of x as $F(x) = \{u : u \text{ is a foot of } x\}$. And define the set of the neighbors of a vertex $v \in N^2(D)$ in $N^1(D)$ to be $F^1(v) = \{u : u \text{ is the neighbor of } v \text{ in } N^1(D)\}$.

Case 1. For each vertex $v \in N^2(D)$, its neighbors in $N^1(D)$ has at least one common foot. That is to say, the set $N^2(D) = \{v_1, v_2, \cdots, v_t\}$ and the neighborhood of v_i in $N^1(D)$ is $F^1(v_i) = \{u_{i,1}, u_{i,2}, \cdots, u_{i,i_l}\}$, where $|F^1(v_i)| \geq 2$, $(i = 1, 2, \cdots, t)$. Then $\bigcap_{a=1}^{i_l} F(u_i, a) \neq \emptyset$ $(i = 1, 2, \cdots, t)$.

In this case, $p(i)\cup q(ii)\cup r(iv)\cup s(iv)\cup G[D]$ (see Figure 1, where the (i), (ii), (iv), (vi) are the subgraphs and p, q, r, s are the numbers of the corresponding subgraphs in G) is a spanning subgraph of G, in which we do not exclude the case that the foots of some vertices are in common. Since each pair of vertices $x, y \in D$ has a proper x-y path in $G[D]$ under the coloring c_D, it suffices to show that $p(i) \cup q(ii) \cup r(iv) \cup s(iv) \cup G[D]$, in which all the vertices in $N^1(D)$ have one common root, has a proper-path coloring using $k + 2$ distinct colors.

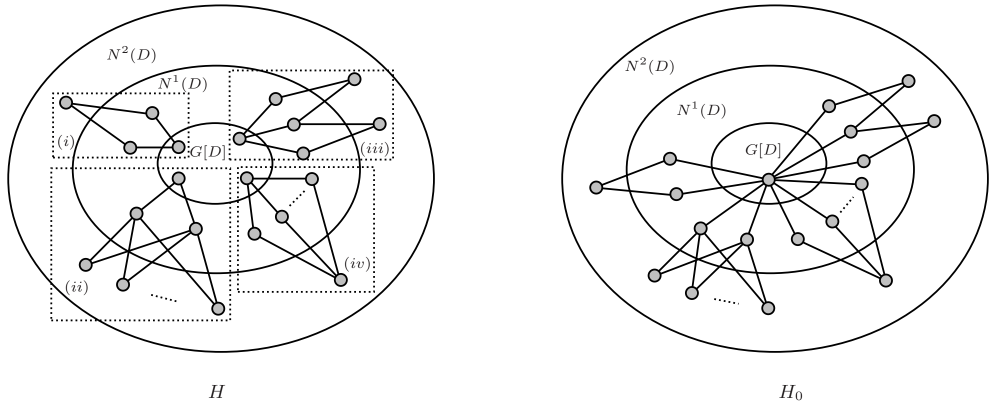

Figure 1: Spanning subgraphs of G

Give an edge-coloring c using colors $\{1, 2, \cdots, k, k+1, k+2\}$ for the above spanning subgraphs H, H_0 of G as follows: for the edges in $G[D]$, we use the proper-path coloring c_D; and for the edges in $(i), (ii), (iii)$ and (iv), color them as depicted in Figure 2. Then for any two vertices $u_i, u_i' \in N^1(D)$, we can find a proper u_i-u_i' path as follows: if $c(u_iv) = c(u_i'v)$, then $u_ixu_jvu_i'$ is a proper u_i-u_i' path; if $c(u_iv) \neq c(u_i'v)$, then u_ivu_i' is a proper u_i-u_i' path. For every pair of vertices $u, v \in N^2(D)$ or $u \in N^1(D), v \in N^2(D)$ or $u \in N^1(D) \cup N^2(D), v \in D$, there exist a proper u-v path under the coloring c as well. This implies that c is a proper-path coloring of the graph H_0 and it follows that $pc(G) \leq pc(H_0) \leq pc(G[D]) + 2$.

Case 2. There exists one vertex $x \in N^2(D)$ whose neighbors in $N^1(D)$ has no common roots. Note that such vertices are not necessarily unique and we can similarly prove the same result as it for x in this theorem.

We give a proper-path coloring c using colors $\{1, 2, \cdots, k := pc(G[D]), k+1, k+2\}$ for a spanning subgraphs H of G as well. Similarly, for the edges in $G[D]$, we still use the proper-path coloring c_D. By the definition of connected two-way two-step dominating sets, x has at least two distinct neighbors in $N^1(D)$ and two edge-disjoined paths connecting to D. This implies that there exist two vertex-disjoint paths, denoted by $P_1 = xu_iv_i, P_2 =$

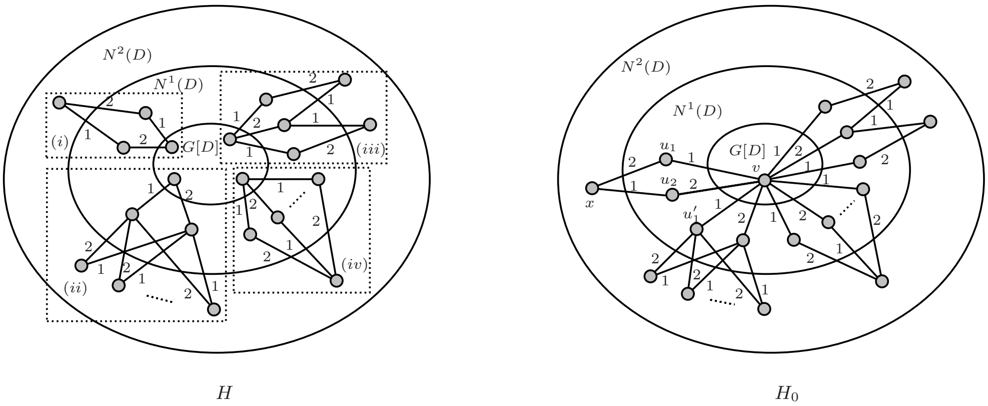

Figure 2: The proper-path coloring for the spanning subgraphs of G

xu_jv_j, where $u_i, u_j \in N^1(D)$ and $v_i, v_j \in D$. We color the edges xu_i with color 1 or color 2 such that $\{1,2\} \subseteq \{c(xu_i) : u_i \in N^1(D)\}$ holds for every vertex $x \in N^2(D)$. And set $c(u_iv_i) \in \{1,2\} \setminus c(xu_i)$. Then for any two vertices $u_i, u_i' \in N^1(D)$, we can find a proper u_i-u_i' path as follows: if $v_i \neq v_i'$, then $u_iv_iP_{ii'}v_i'u_i'$ is a proper u_i-u_i' path, in which $P_{ii'}$ is a proper v_i-v_i' path in $G[D]$; if $v_i = v_i'$ and $c(u_iv_i) = c(u_i'v_i')$, then $u_ixu_jv_i'u_i'$ is a proper u_i-u_i' path, where u_j is a neighbor of x in $N^1(D)$ such that $c(u_ix) \neq c(xu_j)$; if $v_i = v_i'$ and $c(u_iv_i) \neq c(u_i'v_i')$, then $u_iv_iu_i'$ is a proper u_i-u_i' path. Similarly, one can check that for every pair of vertices $u, v \in N^2(D)$ or $u \in N^1(D), v \in N^2(D)$ or $u \in N^1(D) \cup N^2(D), v \in D$, there exists a proper u-v path under the coloring c. It means that c is a proper-path coloring of a spanning subgraph H of G, and then $pc(G) \leq pc(H) \leq pc(G[D]) + 2$.

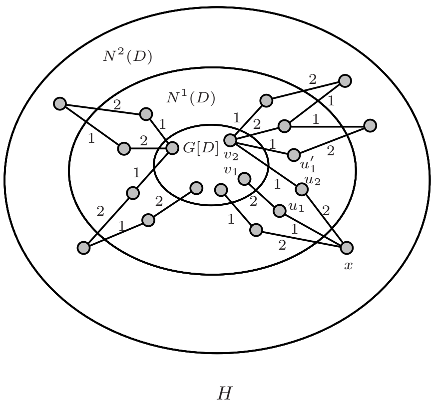

Figure 3: An example for the proper-path coloring of the spanning subgraph

□

Let G be a graph with diameter 2 and minimum degree at least 2. Then there exists one vertex in G which forms a two-way two-step dominating set. And the vertices in

$N^2(D)$ must be contained in the following two structures A and B (see Figure 4), since the diameter of G is 2 and minimum degree is at least 2.

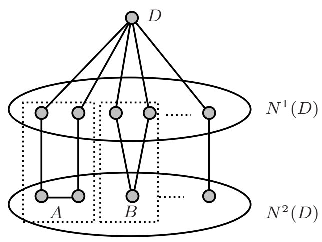

Figure 4: Two structure of the $N^2(D)$ adjacent to $N^1(D)$

Thus, by the proof of the Theorem 3.1, we can easily get the following corollary.

Corollary 3.1. *Let G be a graph with diameter 2 and minimum degree at least 2. Then $pc(G) = 2$.*

A bipartite graph $G(A, B)$ is called a *chain graph*, if the vertices of A can be ordered as $A = (a_1, a_2, \ldots, a_k)$ such that $N(a_1) \subseteq N(a_2) \subseteq \cdots \subseteq N(a_k)$. Applying Theorem 3.1, we can give the proper connection number of a connected chain graph as follows.

Corollary 3.2. *If G is a connected chain graph with minimum degree at least 2, $pc(G) = 2$.*

Proof. Let $G = G(A, B)$ be a connected chain graph, where $A = (a_1, a_2, \ldots, a_k)$, $B = (b_1, b_2, \ldots, b_s)$ such that $b_1 \in N(a_1) \subseteq N(a_2) \subseteq \cdots \subseteq N(a_k)$. Obviously, $N(a_k) = B$ since $G = G(A, B)$ is connected. It is easy to verify that $D = \{b_1\}$ is a connected two-way two-step dominating set of G and $N^1(D) = A,\ N^2(D) = B \backslash \{b_1\}$ (see Figure 5). Applying the result in Theorem 3.1, we obtain that $pc(G) \leq 2$. On the other hand, $pc(G) = 1$ if and only if $G = K_n$ and then $pc(G) \geq 2$. Therefore, $pc(G) = 2$.

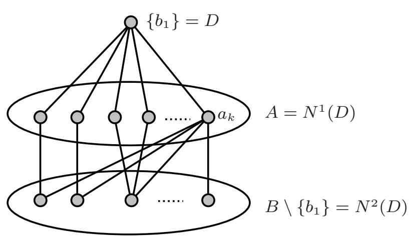

Figure 5: Graph for Corollary 3.2

□

In [4], there is a lemma giving the size of a connected two-way two-step dominating set, which is stated as follows.

Lemma 3.1. *[4] Every connected graph G of order $n \geq 4$ and minimum degree δ has a connected two-way two-step dominating set D of size at most $\frac{3n}{\delta+1} - 2$.*

Then, we get the following corollary.

Corollary 3.3. *Let G be a connected graph of order $n \geq 4$ and minimum degree δ. Then we have*

$$pc(G) \leq \frac{3n}{\delta + 1} - 1.$$

Proof. By Lemma 3.1, the graph G has a connected two-way two-step dominating set D such that $|D| \leq \frac{3n}{\delta+1} - 2$. Since $G[D]$ is connected, $pc(G[D]) \leq |D| - 1 \leq \frac{3n}{\delta+1} - 3$. Together with the result in Theorem 3.1, we obtain that

$$pc(G) \leq pc(G[D]) + 2 \leq \frac{3n}{\delta + 1} - 1.$$

□

Remark 1. This upper bound of proper connection number is not sharp. Further effort is needed to find a sharp upper bound.

Remark 2. If the minimum degree of a graph is at least $\frac{n}{2}$, then the graph is Hamiltonian, and then $pc(G) = 2$. But the corollary shows that if there exists some k such that $\delta = kn$, then the proper connection number can be upper bounded by $\frac{3}{k} - 1$, where $k \leq \frac{1}{2}$.

Remark 3. Since $G[D]$ is a connected subgraph of a graph G, by Proposition 2.1 we know that $pc(G) \leq \chi'(T) + 2$, where T is a spanning tree of $G[D]$.

4 Proper connection number and connected two-way dominating set

The definition of a two-way dominating set (D) implies that every vertex in $V(G) \setminus D$ has at least two edge-disjoint paths connecting to D. Similar with the idea in Section 3, we also obtain the following upper bound for proper connection number.

Theorem 4.1. *If D is a connected two-way dominating set of a graph G, then*

$$pc(G) \leq pc(G[D]) + 2.$$

Proof. We will give a proper-path coloring of a spanning subgraph H of the graph G with $pc(G[D])+2$ colors, which implies this theorem. Let c_D be a proper-path coloring of $G[D]$ using colors $\{3, 4, \cdots, k := pc(G[D]), k+1, k+2\}$.

For any $x \in G \setminus D$, we call a neighbor of x in D a foot of x. Define the set of the foots of x as $F(x) = \{u : u \text{ is a foot of } x\}$. We focus on the case that $|F(x)| = 1$ for every vertex $x \in G \setminus D$. Since D is a connected two-way dominating set, every pendant vertex of G is included in D. Additionally, each pair of vertices $x, y \in D$ has a proper x-y path in $G[D]$ under the coloring c_D and two colors are enough to ensure that a path is proper. Consequently, $p(i) \cup q(ii) \cup G[D]$ (see Figure 6) is a spanning subgraph of G, where we allow that the foots of some vertices are in common. It suffices to show that $p(i) \cup q(ii) \cup G[D]$, in which all the vertices in $G \setminus D$ have a common root, has a proper-path coloring using $k+2$ distinct colors.

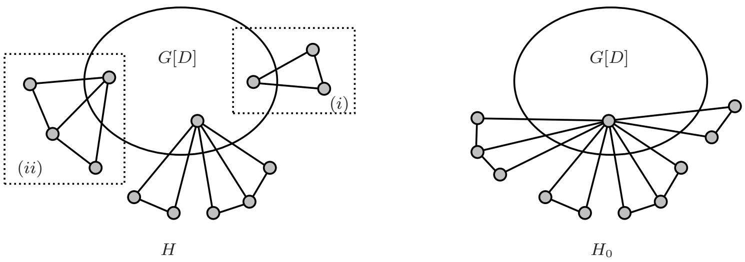

Figure 6: Spanning subgraphs of G

Now we give an edge-coloring c using colors $\{1, 2, \cdots, k, k+1, k+2\}$ for the above spanning subgraphs H, H_0 of G as follows: for the edges in $G[D]$, we use the proper-path coloring c_D; and for the edges in $(i), (ii)$, color them as depicted in Figure 7. Then for any two vertices $u_i, u_j \in G \setminus D$, a proper u_i-u_j path can be found in H_0 under the coloring c as follows: if $u_iu_j \in H_0$, then u_iu_j is a proper u_i-u_j path; if $u_iu_j \notin H_0$, $u_i \in (i)$ and $u_j \in (i)$, then u_ivu_j or $u_ivu_ku_j$ $(u_ku_j \in H_0)$ is a proper u_i-u_j path; if $u_iu_j \notin H_0$, $u_i \in (ii)$ and $u_j \in (ii)$, then u_ivu_j or $u_ivu_ku_j$ $(u_ku_j \in H_0)$ or $u_iu_kvu_j$ $(u_iu_k \in H_0)$ is a proper u_i-u_j path; if $u_iu_j \notin H_0$, $u_i \in (i)$ and $u_j \in (ii)$, then u_ivu_j or $u_iu_kvu_j$ $(u_iu_k \in H_0)$ is a proper u_i-u_j path. One can find a proper u-v path for every pair of vertices $u \in D$ and $v \in G \setminus D$ in similarly way. This implies that c is a proper-path coloring of the graph H_0, and then $pc(G) \leq pc(H_0) \leq pc(G[D]) + 2$.

Next we consider that there exist some vertices $x_i \in G \setminus D$ such that $|F(x_i)| \geq 2$. Let $u_{i_1}, u_{i_2} \in F(x_i)$ for every such vertex x_i. On the basis of the coloring in the above case, color $u_{i_1}x_i, u_{i_2}x_i$ with color 1 if they have not been colored. This provides a proper path

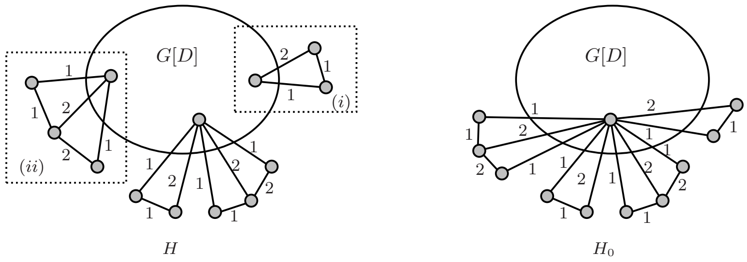

Figure 7: The proper-path coloring for the spanning subgraphs of G

for x_i and every other vertices in G.

Summarizing the above analysis, this theorem holds. □

As consequences of Theorem4.1, the upper bounds for proper connection numbers of interval graphs, asteroidal triple-free graphs, circular arc graphs, threshold graphs, and chain graphs are followed. Before presenting the upper bounds, we state the definitions of all these graphs first.

Definition 4.1. An *intersection graph* of a family of sets F is a graph whose vertices can be mapped to sets in F such that there is an edge between two vertices in the graph if and only if the corresponding two sets in F have a nonempty intersection. An *interval graph* is an intersection graph of intervals on the real line. A *circular arc graph* is an intersection graph of arcs on a circle.

Definition 4.2. An independent triple of vertices x, y, z in a graph G is an *asteroidal triple (AT)*, if between every pair of vertices in the triple, there is a path that does not contain any neighbor of the third. A graph without asteroidal triples is called an asteroidal *triple-free (AT-free) graph*.

Definition 4.3. A graph G is a *threshold graph*, if there exists a weight function $w : V(G) \to R$ and a real constant t such that two vertices $u, v \in V(G)$ are adjacent if and only if $w(u) + w(v) \geq t$.

For a graph G with $\delta(G) \geq 2$, every (connected) dominating set of G is a (connected) two-way dominating set. Next, we will give some upper bounds for the proper connection numbers of the above classes of graphs.

Corollary 4.1. *Let G be a connected non-complete graph with $\delta(G) \geq 2$. Then*

(i) if G is an interval graph, $pc(G) \leq 4$,

(ii) if G is AT-free, $pc(G) \leq 4$,

(iii) if G is a circular arc graph, $pc(G) \leq 4$,

(iv) if G is a threshold graph, $pc(G) = 2$,

Remark. There are four well-known results on the dominating sets of the graphs in Corollary 4.1, which are stated as follows: (i) every interval graph G which is not isomorphic to a complete graph has a dominating path of length at most $diam(G) - 2$, (ii) every AT-free graph G has a dominating path of length at most $diam(G)$, (iii) every circular arc graph G, which is not an interval graph, has a dominating cycle of diameter at most $diam(G)$, (iv) a maximum weight vertex in a connected threshold graph G is a dominating vertex. Together with Theorem 4.1 and Lemma 2.1, we obtain the upper bounds in Corollary 4.1.

Furthermore, we can get a better and sharp bounds on their proper connection numbers by analyzing the structure of the connected interval graph and circular arc graph, which can be stated as follows.

Theorem 4.2. *Let G be a connected interval or circular arc graph with $\delta(G) \geq 2$. Then the proper connection number $pc(G) \leq 3$, and this bound is sharp.*

Proof. The proofs for connected interval graph and circular arc graph are similar, since the circular arc graph is a generalization. So we only give the details for G being a connected interval graph with minimum degree at least 2. We will give a proper-path coloring of G using colors $1, 2, 3$ and as well some graphs arriving at this bound.

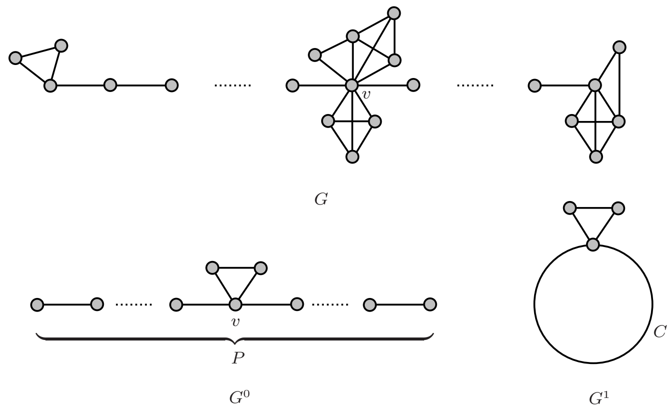

Figure 8: Graphs for Theorem 4.2

Let P be a dominating path of length of $diam(G)-2$ in G. We color the edges of P using colors 1 and 2, alternately. Additionally, by the definition of an interval graph, for every vertex $v \in P$ the subgraph of $G[G \setminus P \cup \{v\}]$ containing v is a maximal clique having at least one common vertex v (see Figure 8). For convenience, we call the vertex v the root of

those maximal cliques. For a fixed vertex $v \in P$, we give a coloring for the edges in those maximal cliques containing v and for any other vertices the same. Let $Q_1, Q_2, \cdots, Q_t$ denote the maximal cliques whose common root is v. Color $u_1v, u_2v, \cdots, u_tv$ with color 3, where $u_i \in Q_i$ and $u_1, u_2, \cdots, u_t$ are not necessarily different. And color all the other edges in $Q_1, Q_2, \cdots, Q_t$ with color 1. One can check that such a coloring is a proper-path coloring of G.

Actually, there are many connected interval graphs (or circular arc graphs) with proper connection number 3, which implies that this bound is sharp. Here we give an infinite class of such graphs. As depicted in Figure 8, G_0 (or G_1), for any vertex $v \in P$(or C), the subgraph of $G[G \setminus P(\text{or } C) \cup \{v\}]$ containing v is a triangle. And for any other vertices in P(or C), those cliques are arbitrary. It is easy to verify that two colors are not enough to make the coloring proper for G. $\square$

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